3.4 Symétries et lois de conservation

Dans cette section, on supposera que la densité lagrangienne ne dépend pas explicitement de (x_μ). On supposera aussi que les équations de mouvement (donc de l'action) restent inchangées lors d'une transformation (continue) infinitésimale définit par,

$$\begin{cases} x_\mu \longrightarrow x'_\mu = x_\mu + \delta x_\mu \\ \phi(x_\mu) \longrightarrow \phi'(x'_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) \end{cases} \tag{3.55}$$

Avec,

$$\begin{cases} x_\mu \longrightarrow \text{position spatio-temporelle (coordonnées)} \\ \delta x_\mu \longrightarrow \text{variation infinitisimale (deplacement l'espace et dans le temps)} \\ \phi(x_\mu) \longrightarrow \text{champ scalaire (variable)} \\ \delta\phi(x_\mu) \longrightarrow \text{variation de phase (dûe à une rotation)} \end{cases}$$

3.4.1 Exemple de transformation

Transformation spatio-temporelle

Une transformation spatio-temporelle est définit par

$$\begin{cases} x_\mu \longrightarrow x'_\mu = x_\mu + a_\mu \ , \quad (a_\mu = \delta x_\mu) \\ \phi(x_\mu) \longrightarrow \phi'(x'_\mu) = \phi(x_\mu) \ , \quad (\delta\phi(x_\mu) = 0) \end{cases} \tag{3.56}$$

Avec a_μ représente le quadri-vecteur déplacement dans l'espace-temps.
D'après la transformation infinitésimale donnée dans l'équation (3.56),

$$\phi'(x'_\mu) = \phi'(x_\mu + a_\mu) = \phi(x_\mu) \tag{3.57}$$

donc,

$$\phi'(x_\mu + a_\mu) = \phi(x_\mu) \tag{3.58}$$

Transformation de phase globale $(\phi(x_\mu) \neq \phi^*(x_\mu))$

Cette transformation est donnée par,

$$\begin{cases} x_\mu \longrightarrow x'_\mu = x_\mu \ , \quad (\delta x_\mu = 0) \\ \phi(x_\mu) \longrightarrow \phi'(x'_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) = e^{-iq\theta(x_\mu)}\phi(x_\mu) \end{cases} \tag{3.59}$$

Avec $\theta(x_\mu)$ est un scalaire réel.
D'après la transformation infinitésimale donnée dans l'équation (3.59),

$$\phi^{'}(x^{'}_\mu) = \phi^{'}(x_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) = e^{-iq\theta(x_\mu)}\phi(x_\mu) \tag{3.60}$$

donc,

$$\phi^{'*}(x_\mu) = e^{+iq\theta(x_\mu)}\phi^*(x_\mu) \tag{3.61}$$

Transformation de phase locale $(\phi(x_\mu) = \phi^*(x_\mu))$

Cette transformation est donnée par,

$$\begin{cases} x_\mu \longrightarrow x^{'}_\mu = x_\mu \ , \quad (\delta x_\mu = 0) \\ \phi(x_\mu) \longrightarrow \phi^{'}(x^{'}_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) = e^{-iq\theta(x_\mu)}\phi(x_\mu) \end{cases} \tag{3.62}$$

Avec $\theta(x_\mu)$ est un scalaire réel.
D'après la transformation infinitésimale donnée dans l'équation (3.62),

$$\phi^{'}(x^{'}_\mu) = \phi^{'}(x_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) = e^{-iq\theta(x_\mu)}\phi(x_\mu) \tag{3.63}$$

donc,

$$\phi^{'*}(x_\mu) = e^{+iq\theta(x_\mu)}\phi(x_\mu) \tag{3.64}$$

3.4.2 Théorème de Noether

Énoncé

Pour toute transformation continue de l'action S, il existe un courant J_μ vérifiant l'équation

$$\partial_\mu J_\mu = 0 \tag{3.65}$$

Ceci implique qu'il existe une charge qui se conserve, définit par

$$Q = \int \rho \, d^3x \tag{3.66}$$

Démonstration

On dit que les équations de mouvement sont invariantes si l'action S est stationnaire

$$\delta S = S^{'} - S \simeq 0 \tag{3.67}$$

On a

$$S = \int d^4x\, \mathcal{L}(\phi, \partial_\mu \phi) \Rightarrow S^{'} = \int d^4x^{'}\, \mathcal{L}(\phi^{'}, \partial^{'}_\mu \phi^{'}) \tag{3.68}$$

Avec $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu \phi)$ (la densité lagrangienne ne dépend pas explicitement de x_μ).
Considérons des transformations infinitésimales de la forme,

$$\begin{cases} x_\mu \longrightarrow x^{'}_\mu = x_\mu + \delta x_\mu \\ \phi(x_\mu) \longrightarrow \phi^{'}(x^{'}_\mu) = \phi(x_\mu) + \delta\phi(x_\mu) \end{cases} \tag{3.69}$$

où

$$\delta\phi(x) = \phi^{'}(x^{'}) - \phi(x) \tag{3.70}$$

Avec $\delta\phi(x_\mu)$ représente la variation du champ dû à la fois à la transformation du champ (variable) et la transformation de la coordonnées (x_μ).
Donc, la variation en un point fixe de l'espace à 4-dimensions est donnée par

$$\delta_o\phi(x) = \phi^{'}(x) - \phi(x) \ , \quad \text{pour } x^{'} = x \tag{3.71}$$

Le lien entre les dérivées d'espace-temps est donné par

$$d^4x^{'} = [1 + \partial_\mu(\delta x_\mu)]d^4x \tag{3.72}$$

Cherchons maintenant la relation entre la variation champ en deux points différents $\delta\phi$ et la variation du champ en un point fixe $\delta_o\phi$.
La variation du champ en deux point différents est donnée par

$$\delta\phi(x) = \phi^{'}(x^{'}) - \phi(x) = \phi^{'}(x^{'}) - \phi^{'}(x) + \phi^{'}(x) - \phi(x) \tag{3.73}$$

$$\delta\phi(x) = \phi^{'}(x) + (\partial_\nu\phi)\delta x_\nu - \phi^{'}(x) + \delta_o\phi(x) \tag{3.74}$$

Avec

$$\phi^{'}(x^{'}) = \phi^{'}(x_\mu + \delta x_\mu) = \phi^{'}(x_\mu) + (\partial_\nu\phi)\delta x_\nu = \phi^{'}(x) + (\partial_\nu\phi)\delta x_\nu \tag{3.75}$$

Donc,

$$\delta\phi(x) = \delta_o\phi(x) + (\partial_\nu\phi)\delta x_\nu \tag{3.76}$$

Calculons le terme $\partial'_\mu\phi'$
On a

$$\partial'_\mu\phi'(x') = \partial'_\mu(\phi + \delta\phi) = \frac{\partial}{\partial x'_\mu}(\phi + \delta\phi) \tag{3.77}$$

$$= \frac{\partial}{\partial x_\nu}\frac{\partial x_\nu}{\partial x'_\mu}(\phi + \delta\phi) = \frac{\partial}{\partial x_\nu}(\phi + \delta\phi)\frac{\partial x_\nu}{\partial x'_\mu} \tag{3.78}$$

On a aussi

$$x'_\nu = x_\nu + \delta x_\nu \Rightarrow x_\nu = x'_\nu - \delta x_\nu \tag{3.79}$$

Donc,

$$\frac{\partial x_\nu}{\partial x'_\mu} = \frac{\partial x'_\nu}{\partial x'_\mu} + \frac{\partial}{\partial x'_\mu}(\delta x_\nu) \tag{3.80}$$

Finalement on trouve que,

$$\frac{\partial x_\nu}{\partial x'_\mu} = \delta_{\mu\nu} - \partial_\mu(\delta x_\nu) \tag{3.81}$$

En remplaçant l'équation (3.81) dans l'équation (3.77), on trouve

$$\partial'_\mu\phi'(x') = \frac{\partial}{\partial x_\nu}(\phi + \delta\phi)\frac{\partial x_\nu}{\partial x'_\mu} \tag{3.82}$$

$$= \left(\frac{\partial\phi}{\partial x_\nu} + \frac{\partial}{\partial x_\nu}(\delta\phi)\right)\left(\delta_{\mu\nu} - \partial_\mu(\delta x_\nu)\right) \tag{3.83}$$

$$= \left(\partial_\nu\phi + \partial_\nu(\delta\phi)\right)\left(\delta_{\mu\nu} - \partial_\mu(\delta x_\nu)\right) \tag{3.84}$$

$$= (\partial_\nu\phi)\delta_{\mu\nu} - (\partial_\nu\phi)\partial_\mu(\delta x_\nu) + \partial_\nu(\delta\phi)\delta_{\mu\nu} - \partial_\nu(\delta\phi)\partial_\mu(\delta x_\nu) \tag{3.85}$$

$$\partial'_\mu\phi'(x') = (\partial_\mu\phi) - (\partial_\nu\phi)\partial_\mu(\delta x_\nu) + \partial_\mu(\delta\phi) \tag{3.86}$$

On néglige le terme $\partial_\nu(\delta\phi)\partial_\mu(\delta x_\nu)$, car c'est un terme d'ordre supérieur.
La densité lagrangienne ne dépend pas explicitement de $x_\mu \Rightarrow \mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$
Donc

$$\mathcal{L}(\phi', \partial'_\mu\phi') = \mathcal{L}(\phi + \delta\phi, (\partial_\mu\phi) - (\partial_\nu\phi)\partial_\mu(\delta x_\nu) + \partial_\mu(\delta\phi)) \tag{3.87}$$

$$= \mathcal{L}(\phi, \partial_\mu\phi) + \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\left[\partial_\mu(\delta\phi) - (\partial_\nu\phi)\partial_\mu(\delta x_\nu)\right] \tag{3.88}$$

Donc, on trouve

$$\mathcal{L}(\phi', \partial'_\mu \phi') = \mathcal{L}(\phi, \partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial_\nu \phi)\partial_\mu(\delta x_\nu) \tag{3.89}$$

En remplaçant l'équation (3.72) dans l'équation (3.67), on trouve

$$\delta S = \int d^4x' \, \mathcal{L}(\phi', \partial'_\mu \phi') - \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi) \simeq 0 \tag{3.90}$$

$$= \int [1 + \partial_\mu(\delta x_\mu)] d^4x \, \mathcal{L}(\phi', \partial'_\mu \phi') - \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi) \simeq 0 \tag{3.91}$$

$$\delta S = \int [\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) + \partial_\mu(\delta x_\mu)\mathcal{L}] d^4x \simeq 0 \tag{3.92}$$

Calculons le terme: $\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi)$

$$\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) = \mathcal{L}(\phi, \partial_\mu \phi) + \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial_\nu \phi)\partial_\mu(\delta x_\nu) - \mathcal{L}(\phi, \partial_\mu \phi)$$

$$\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi) - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial_\nu \phi)\partial_\mu(\delta x_\nu) \tag{3.93}$$

D'après les équations d'Euler-Lagrange,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

Donc

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \tag{3.94}$$

On a aussi

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \right) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi)$$

Donc,

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\partial_\mu(\delta\phi) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)\delta\phi \tag{3.95}$$

En remplaçant les équations (3.94) et (3.95) dans l'équation (3.93), on trouve

$$\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)\delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)\delta\phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial_\nu \phi)\partial_\mu(\delta x_\nu)$$

$$\mathcal{L}(\phi',\partial'_\mu\phi') - \mathcal{L}(\phi,\partial_\mu\phi) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu(\delta x_\nu) \tag{3.96}$$

On a

$$\delta\phi = \delta_o\phi + (\partial_\nu\phi)\delta x_\nu$$

Donc

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\delta_o\phi + (\partial_\nu\phi)(\delta x_\nu))\right)$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)(\delta x_\nu)\right) \tag{3.97}$$

Calculons le terme $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)(\delta x_\nu)\right)$:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)(\delta x_\nu)\right) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu\left((\partial_\nu\phi)\right)(\delta x_\nu) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu\left(\delta x_\nu\right)$$

En négligeant les termes d'ordre supérieur, on trouve

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)(\delta x_\nu)\right) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu\left(\delta x_\nu\right) \tag{3.98}$$

Donc,

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu\left(\delta x_\nu\right) \tag{3.99}$$

En remplaçant l'équation (3.99) dans l'équation (3.96), on trouve

$$\mathcal{L}(\phi',\partial'_\mu\phi') - \mathcal{L}(\phi,\partial_\mu\phi) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu(\delta x_\nu)$$

$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu\left(\delta x_\nu\right) - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(\partial_\nu\phi)\partial_\mu\left(\delta x_\nu\right)$$

Donc,

$$\mathcal{L}(\phi',\partial'_\mu\phi') - \mathcal{L}(\phi,\partial_\mu\phi) = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_o\phi\right) + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu)$$

Calculons le terme $\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu)$:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)(\partial_\nu\phi)(\delta x_\nu) = \frac{\partial\mathcal{L}}{\partial\phi}(\partial_\nu\phi)(\delta x_\nu) = \frac{\partial\mathcal{L}}{\partial x_\mu}\frac{\partial x_\mu}{\partial\phi}\frac{\partial\phi}{\partial x_\nu}\delta x_\nu$$

$$= \frac{\partial \mathcal{L}}{\partial x_\mu}\frac{\partial x_\mu}{\partial \partial x_\nu}\delta x_\nu = \frac{\partial \mathcal{L}}{\partial x_\mu}\delta_{\mu\nu}\delta x_\nu = \partial_\mu \mathcal{L}\ \delta x_\mu$$

Finalement, on trouve

$$\mathcal{L}(\phi', \partial'_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_o \phi \right) + \partial_\mu \mathcal{L}\ \delta x_\mu \tag{3.100}$$

La variation de l'action dans l'équation (3.92) devient,

$$\delta S = \int \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_o \phi \right) + \partial_\mu \mathcal{L}\ \delta x_\mu + \partial_\mu(\delta x_\mu)\mathcal{L} \right] d^4x \simeq 0$$

On a

$$\partial_\mu \mathcal{L}\ \delta x_\mu + \partial_\mu(\delta x_\mu)\mathcal{L} = \partial_\mu(\mathcal{L}\,\delta x_\mu)$$

Alors,

$$\delta S = \int \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_o \phi \right) + \partial_\mu(\mathcal{L}\,\delta x_\mu) \right] d^4x \simeq 0$$

$$\delta S = \int \partial_\mu \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_o \phi \right) + \mathcal{L}\,\delta x_\mu \right] d^4x \simeq 0$$

$$\Rightarrow \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_o \phi + \mathcal{L}\,\delta x_\mu \right] = 0$$

Cette dernière équation peut être écrite sous la forme

$$\partial_\mu J_\mu = 0$$

Avec,

$$J_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_o \phi + \mathcal{L}\,\delta x_\mu \longrightarrow \text{Courant de Noether}$$

Exercice 12 :

1. Montrer que la densité lagrangienne du champ scalaire complexe libre est invariante dans la transformation de phase globale suivante,

$$\begin{cases} \phi(x) \longrightarrow \phi'(x) = e^{i\theta}\phi(x) \\ \phi^*(x) \longrightarrow \phi'^*(x) = e^{-i\theta}\phi^*(x) \end{cases}$$

   où θ est un réel indépendants de x_μ.

2. Quels sont les courant et charge qui se conservent?

3.5 Tenseur Énergie-Impulsion du champ scalaire

Étant donner que densité lagrangienne $\mathcal{L}$ ne dépend pas explicitement du quadri-vecteur position x_μ, sa dérivée par rapport à x_μ est donnée par

$$\partial_\mu \mathcal{L} = \partial_\mu \mathcal{L}(\phi, \partial_\mu \phi) \quad \text{où} \quad \partial_\mu = \frac{\partial}{\partial x_\mu} \tag{3.101}$$

Donc,

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x_\mu} \tag{3.102}$$

On a,

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x_\mu} = \frac{\partial \mathcal{L}}{\partial \phi}\frac{\partial \phi}{\partial x_\mu} + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\frac{\partial(\partial_\nu \phi)}{\partial x_\mu} \tag{3.103}$$

Or, d'après l'équation d'Euler-Lagrange on a

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\right) = 0 \quad \Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} = \partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\right) \quad \text{pour} \qquad \mu = \nu \tag{3.104}$$

Donc,

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x_\mu} = \partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\right)\partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\partial_\mu\left(\partial_\nu \phi\right) \tag{3.105}$$

On pose,

$$\partial_\mu\left(\partial_\nu \phi\right) = \partial_\nu\left(\partial_\mu \phi\right) \tag{3.106}$$

On trouve,

$$\partial_\mu \mathcal{L} = \partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\right)\partial_\mu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\partial_\nu\left(\partial_\mu \phi\right) = \partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\partial_\mu \phi\right) \tag{3.107}$$

Le terme $\partial_\mu \mathcal{L}$ peut être écrit aussi sous la forme:

$$\partial_\mu \mathcal{L} = \frac{\partial \mathcal{L}}{\partial x_\mu} = \frac{\partial \mathcal{L}}{\partial x_\nu}\frac{\partial x_\nu}{\partial x_\mu} = \left(\partial_\nu \mathcal{L}\right)\delta_{\mu\nu} = \partial_\nu\left(\mathcal{L}\delta_{\mu\nu}\right) \tag{3.108}$$

Finalement, en comparant les équation (3.107) et (3.108), on trouve

$$\partial_\mu \mathcal{L} = \partial_\nu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\partial_\mu \phi\right) = \partial_\nu\left(\mathcal{L}\delta_{\mu\nu}\right) \tag{3.109}$$

Donc,

$$\partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi - \mathcal{L} \delta_{\mu\nu} \right) = 0 \tag{3.110}$$

Maintenant, si on remplace ν par μ

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta_{\mu\nu} \right) = 0 \tag{3.111}$$

Cette dernière équation peut être réécrite sous la forme suivante,

$$\partial_{\mu\nu} T_{\mu\nu} = 0 \text{ avec } T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta_{\mu\nu} \tag{3.112}$$

Où $T_{\mu\nu}$ représente le tenseur énergie-impulsion du champ scalaire.