Université de Djilali Bounaama
Faculté des Sciences et de la Technologie
Département de Mathémathique et Informatique

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Module:Analyse complexe
Deuxmière année maths

Corrigé de Série N=02

Corrigé 1. 1.$f(z) = x^2 - y^2 + x + 1 + i(2xy + y)$

$$\begin{aligned}\frac{\partial u}{\partial x} &= 2x + 1 = \frac{\partial v}{\partial y}.\\ \frac{\partial u}{\partial y} &= -2y = -\frac{\partial v}{\partial x}\end{aligned}$$

La fonction est donc bien holomorphe, sa dérivée est

$$f^{'}(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + 1 + 2iy$$

2.$f(z) = 2x^2 + 2y^2 - x + i(2xy - y)$

$$\begin{aligned}\frac{\partial u}{\partial x} &= 4x - 1 = \frac{\partial v}{\partial y}.\\ \frac{\partial u}{\partial y} &= 4y \neq -\frac{\partial v}{\partial x}\end{aligned}$$

La fonction n'est donc pas holomorphe.
3.$f(z) = z^2 + z + \ln z (\Re z \succ 0)$

$$\frac{\partial f}{\partial \overline{z}} = 0 + 0 + 0 = 0$$

La fonction est donc holomorphe sauf en $z = 0$ *, sa dérivée est*

$$f^{'}(z) = \frac{\partial f}{\partial z} = 2z + 1 + \frac{1}{z}$$

4.$f(z) = z^2 + \overline{z} + \cos z$

$$\frac{\partial f}{\partial \overline{z}} = 1 \neq 0$$

La fonction n'est donc pas holomorphe

Corrigé 2. $1.u_1(x,y) = 3x + 1$

$$\begin{aligned}
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad &\Rightarrow \quad 3 = \frac{\partial v}{\partial y} \Rightarrow v = \int 3dy = 3y + A(x). \\
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad &\Rightarrow \quad 0 = \frac{-\partial A}{\partial x} \Rightarrow A(x) = \int 0dx = C^{te} \\
&\Rightarrow \quad f(x,y) = u + iv = 3x + 1 + i3y + C^{te}
\end{aligned}$$

$2.u_2(x,y) = x^2 - y^2 + 2x + 1$

$$\begin{aligned}
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad &\Rightarrow \quad 2x + 2 = \frac{\partial v}{\partial y} \Rightarrow v = \int (2x+2)dy = 2xy + 2y + A(x). \\
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad &\Rightarrow \quad -2y = -2y - \frac{\partial A}{\partial x} \Rightarrow A(x) = \int 0dx = C^{te} \\
&\Rightarrow \quad f(x,y) = u + iv = x^2 - y^2 + 2x + 1 + i2(xy + y) + C^{te}
\end{aligned}$$

$3.u_3(x,y) = -2xy$

$$\begin{aligned}
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad &\Rightarrow \quad -2y = \frac{\partial v}{\partial y} \Rightarrow v = \int -2ydy = -y^2 + A(x). \\
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad &\Rightarrow \quad -2x = -\frac{\partial A}{\partial x} \Rightarrow A(x) = \int 2xdx = x^2 + C^{te} \\
&\Rightarrow \quad f(x,y) = u + iv = -2x + i(x^2 - y^2) + C^{te}
\end{aligned}$$

$4.v_1(x,y) = e^x \sin y$

$$\begin{aligned}
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad &\Rightarrow \quad \frac{\partial u}{\partial x} = e^x \cos y \Rightarrow u = \int e^x \cos y dx = e^x \cos y + A(y). \\
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad &\Rightarrow \quad -e^x siny + \frac{\partial A}{\partial y} = -e^x \sin y \Rightarrow A(y) = \int 0dy = C^{te} \\
&\Rightarrow \quad f(x,y) = e^x \cos y + ie^x \sin y + C^{te} = e^x(\cos y + i \sin y) + C^{te} \\
&\Rightarrow \quad f(z) = e^z + C^{te}
\end{aligned}$$

$5.v_2(x,y) = 2x + 2xy$

$$\begin{aligned}
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad &\Rightarrow \quad \frac{\partial u}{\partial x} = 2x \Rightarrow u = \int 2xdx = x^2 + A(y). \\
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad &\Rightarrow \quad \frac{\partial A}{\partial y} = -2 - 2y \Rightarrow A(y) = \int (-2-2y)dy = -2y - y^2 + C^{te} \\
&\Rightarrow \quad f(x,y) = x^2 - y^2 - 2y + i2x + i2xy + C^{te} \\
&\Rightarrow \quad f(z) = z^2 + 2iz + C^{te}
\end{aligned}$$

$6.v_3(x,y) = x^2 - y^2$

$$\begin{aligned}
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} &\Rightarrow \frac{\partial u}{\partial x} = -2y \Rightarrow u = \int 2y dx = -2xy + A(y). \\
\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} &\Rightarrow -2x + \frac{\partial A}{\partial y} = -2x \Rightarrow A(y) = \int 0 dy = C^{te} \\
&\Rightarrow f(x,y) = -2xy + i(x^2 - y^2) + C^{te} \\
&\Rightarrow f(z) = i(z^2) + C^{te}
\end{aligned}$$

Corrigé 3. *posons $\lambda = a + ib$ et utilisant les équations de Cauchy -Reiman pour trouver a et b.*
$1.f(z) = x + i\lambda y$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Longrightarrow \begin{cases} a = 1 \\ -b = 0 \end{cases} \Longrightarrow (a = 1, b = 0) \Longrightarrow \lambda = 0$$

$2.f(z) = x^2 - y^2 + \lambda x + i(\lambda y + 2xy)$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Longrightarrow \begin{cases} 2x + a = a + 2x \\ -2y - b = -b - 2y \end{cases}$$

$\Longrightarrow$ *(a et b quelconque): Les λ sont tous les points du plan complexe.*
$3.f(z) = Re(\lambda)Re(z) + i[Im(\lambda) + 1]Im(z)$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases} \Longrightarrow \begin{cases} a = b + 1 \\ 0 = 0 \end{cases}$$

$\Longrightarrow (b = a - 1)$ *Les λ sont les points d'une droite.*

Corrigé 4. $1.u(x,y) = xy$

$$\begin{cases} \frac{\partial u}{\partial x}(x,y) = y \\ \frac{\partial^2 u}{\partial x^2}(x,y) = 0 \end{cases} \Longrightarrow \begin{cases} \frac{\partial u}{\partial y}(x,y) = x \\ \frac{\partial^2 u}{\partial y^2}(x,y) = 0 \end{cases}$$

Donc

$$\Delta u = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0$$

d'où, u est harmonique.
$2.u(x,y) = x^2 - y^2 + xy$

$$\begin{cases} \frac{\partial u}{\partial x}(x,y) = 2x + y \\ \frac{\partial^2 u}{\partial x^2}(x,y) = 2 \end{cases} \Longrightarrow \begin{cases} \frac{\partial u}{\partial y}(x,y) = -2y + x \\ \frac{\partial^2 u}{\partial y^2}(x,y) = -2 \end{cases}$$

Donc

$$\Delta u = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0$$

d'où, u est harmonique.

$3.u(x,y) = e^x x \cos y - e^x y \sin y$

$$\begin{cases} \frac{\partial u}{\partial x}(x,y) &= e^x \cos y(x+1) - ye^x \sin y \\ \frac{\partial^2 u}{\partial x^2}(x,y) &= e^x(-y \sin y + 2\cos y + x \cos y) \end{cases}$$

$\Longrightarrow$

$$\begin{cases} \frac{\partial u}{\partial y}(x,y) &= -xe^x \sin y - e^x(\sin y + y \cos y) \\ \frac{\partial^2 u}{\partial y^2}(x,y) &= -e^x(-y \sin y + 2\cos y + x \cos y) \end{cases}$$

Donc

$$\Delta u = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0$$

d'où, u est harmonique.