Correction :

Soit l'EDP : $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 0$ ... (*)

On a $\Delta = b^2 - 4ac = 4x^2y^2 - 4x^2y^2 = 0$

alors l'équation caractéristique est donnée par

$\frac{dy}{dx} = \frac{b - \sqrt{\Delta}}{2a} = \frac{2xy}{2x^2} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x} + k$

$\Rightarrow \ln y = \ln x + k$ c-à-d, $\ln \frac{y}{x} = k$, Alors

$\frac{y}{x} = e^k = C = \varphi(x,y)$, d'où : $\boxed{\varphi(x,y) = \frac{y}{x}}$

et on peut prendre $\psi(x,y) = x$, telle que

$$J(\varphi,\psi) = \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ 1 & 0 \end{vmatrix} = -\frac{1}{x} \neq 0$$

Cherchons la forme canonique de l'EDP (*)

On a : d'après la règle de chaîne,

1/ $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \varphi}\left(\frac{-y}{x^2}\right) + \frac{\partial u}{\partial \psi} = -\frac{y}{x^2}\frac{\partial u}{\partial \varphi} + \frac{\partial u}{\partial \psi}$

2/ $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x}\left[-\frac{y}{x^2}\frac{\partial u}{\partial \varphi} + \frac{\partial u}{\partial \psi}\right]$

$= \frac{2y}{x^3}\frac{\partial u}{\partial \varphi} - \frac{y}{x^2}\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial \varphi}\right] + \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial \psi}\right]$

$= \frac{2y}{x^3}\frac{\partial u}{\partial \varphi} - \frac{y}{x^2}\left[\frac{\partial^2 u}{\partial \varphi^2}\left(-\frac{y}{x^2}\right) + \frac{\partial^2 u}{\partial \varphi \partial \psi}\right] + \frac{\partial^2 u}{\partial \varphi \partial \psi}\left(\frac{-y}{x^2}\right) + \frac{\partial^2 u}{\partial \psi^2}$

$= \frac{2y}{x^3}\frac{\partial u}{\partial \varphi} + \frac{y^2}{x^4}\frac{\partial^2 u}{\partial \varphi^2} - \frac{2y}{x^2}\frac{\partial^2 u}{\partial \varphi \partial \psi} + \frac{\partial^2 u}{\partial \psi^2}$

3/ $\frac{\partial u}{\partial y} = \frac{1}{x}\frac{\partial u}{\partial \varphi}$

4/ $\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{1}{x}\frac{\partial u}{\partial \varphi}\right] = \frac{1}{x}\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial \varphi}\right] = \frac{1}{x}\left[\frac{\partial^2 u}{\partial \varphi^2}\cdot\frac{1}{x}\right]$

$= \frac{1}{x^2}\frac{\partial^2 u}{\partial \varphi^2}$.

5/ $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y}\right] = \frac{1}{\partial x}\left[\frac{1}{x}\frac{\partial u}{\partial \varphi}\right]$

$= -\frac{1}{x^2}\frac{\partial u}{\partial \varphi} + \frac{1}{x}\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial \varphi}\right]$

$= -\frac{1}{x^2}\frac{\partial u}{\partial \varphi} + \frac{1}{x}\left[\frac{\partial^2 u}{\partial \varphi^2}\cdot\left(\frac{-y}{x^2}\right) + \frac{\partial^2 u}{\partial \psi \partial \varphi}\right]$

$= -\frac{1}{x^2}\frac{\partial u}{\partial \varphi} - \frac{y}{x^3}\frac{\partial^2 u}{\partial \varphi^2} + \frac{1}{x}\frac{\partial^2 u}{\partial \varphi \partial \psi}$

Alors la forme standard de l'EDP (*) est la suivante :

(*) $\Longleftrightarrow x^2 \frac{\partial^2 u}{\partial \psi^2} = 0$, on sait que $x \neq 0$

alors : (*) $\Longleftrightarrow \frac{\partial^2 u}{\partial \psi^2} = 0$.

La solution de l'EDP (*) est donnée par :

$u(\varphi, \psi) = f(\varphi)\psi + g(\varphi)$

donc : $u(x,y) = F\left(\frac{y}{x}\right)x + G\left(\frac{y}{x}\right)$

où F et G sont deux fonctions différentiables arbitraires

— fin —