Math 1 Correction (Serie d'exercice N°: 3.)

Math I

Fonctions reelles - limites - continuité - Derivation.

Exo 1

① $f(x) = \dfrac{\sqrt{4-3x^2}}{x}$

$D_f = \{x \in \mathbb{R} / \ 4-3x^2 \geq 0 \text{ et } x \neq 0\}$

$D_f = \{x \in \mathbb{R} / (2-\sqrt{3}x)(2+\sqrt{3}x) \geq 0 \text{ et } x \neq 0\}$

$-2/\sqrt{3}$ $\quad$ $2/\sqrt{3}$

$- \quad 0 \quad + \quad 0 \quad -$

$D_f = [-\frac{2}{\sqrt{3}}, 0[\cup]0, \frac{2}{\sqrt{3}}[$

② $g(x) = \dfrac{\ln(4-|x-1|)}{\sqrt[3]{2-x}}$

$g(x) = \dfrac{\ln(4-|x-1|}{e^{1/3 \ln(2-x)}}$

$D_g = \{x \in \mathbb{R} / \ 4-|x-1| > 0 \text{ et } 2-x > 0$

$4-|x-1| > 0 \iff |x-1| < 4$

$-4 < x-1 < 4$

$-3 < x < 5$

$x \in]-3, 5[$

$\sqrt[n]{x}$: n pair $\rightarrow D_g = \mathbb{R}^+$; n impair $\rightarrow D_g = \mathbb{R}$

$\sqrt[3]{2-x}$ est defini sur $\mathbb{R}$ car 3 est impair.

$-\infty \quad -3 \quad 2 \quad 5 \quad +\infty$

$\boxed{D_g =]-3, 2[\cup]2, 5[}$

Exo 2

①

$D_f = \mathbb{R}$

$\forall x \in \mathbb{R}, -x \in \mathbb{R}$ et

$f(-x) = \cos(-4x) + e^{-(-x)^2}$

$= \cos(4x) + e^{-x^2}$

$= f(x)$

f est la fct paire

② $D_g = \mathbb{R} - \{\frac{\pi}{2} + k\pi\}$

$\forall x \in D_g, -x \in D_g$ et

$g(-x) = (-x)^3 + \sin(-x) + \tan(-x)$

$= -x^3 - \sin x - \tan(x)$

$= -(x^3 + \sin x + \tan(x))$

$= -f(x)$

③ $h(x) = \dfrac{\ln(1+x)}{|x|}$

$D_h = \{x \in \mathbb{R} / \ 1+x > 0 \text{ et } x \neq 0\}$

$D_h =]-1, 0[\cup]0, +\infty[$

on ne peut pas etudier la parité de h car le domaine de def n'est pas symetrique par rapport à zero

S.L.

EX03 :

①
$$\lim_{x\to 2} \frac{x^4-16}{x-2} = \lim_{x\to 2} \frac{(x^2-4)(x^2+4)}{x-2} = \frac{0}{0} \text{ F.I.}$$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)(x^2+4)}{\cancel{(x-2)}}$$

$$= \lim_{x\to 2} (x+2)(x^2+4)$$

$$= 32$$

②
$$\lim_{x\to+\infty} \sqrt{x^2+1} - x = \infty - \infty \text{ F.I.}$$

$$= \lim_{x\to+\infty} \frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{\sqrt{x^2+1}+x}$$

$$= \lim_{x\to+\infty} \frac{x^2+1-x^2}{\sqrt{x^2+1}+x} = \lim_{x\to+\infty} \frac{1}{|x|\sqrt{1+\frac{1}{x^2}}+x}$$

$$= \lim_{x\to+\infty} \frac{1}{x\left(\sqrt{1+\frac{1}{x^2}}+1\right)} = 0$$

③
$$\lim_{x\to 0} \frac{\sqrt{1-x^2}-1}{x}$$

on pose $g(x) = \sqrt{1-x^2}$

$$g'(x) = \frac{-2x}{2\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$$g(0) = 1 \ , \ g'(0) =$$

$$\lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = \lim_{x\to 0} \frac{\sqrt{1-x^2}-1}{x-0}$$

$$= g'(0)$$

$$= 0$$

$$\lim \frac{1-\sqrt{1-x^2}}{x} = 0.$$

4)
$$\lim_{\substack{x\to 2\\ >}} \frac{|x-2|}{x-2} = 1$$

$$\lim_{\substack{x\to 2\\ <}} \frac{-(x-2)}{x-2} = -1.$$

Don $\lim_{x\to 2} \frac{|x-2|}{x-2}$ n'existe pas.

5)
$$\lim_{x\to 0^+} \sqrt{x}\ \sin\left(\frac{\pi}{x}\right)$$

on a : $\forall x \in \mathbb{R}$

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$$

$$-\sqrt{x} \leq \sqrt{x}\sin\left(\frac{\pi}{x}\right) \leq \sqrt{x}$$

$(\sqrt{x} > 0)$

$$0 \leq \lim_{x\to 0} \sqrt{x}\sin\left(\frac{\pi}{x}\right) \leq 0$$

$$\lim_{x\to 0} \sqrt{x}\ \sin\left(\frac{\pi}{x}\right) = 0 \quad \text{Th. J}$$

theoreme de gendarme.

⑥ $\forall x \in \mathbb{R}$

$$-1 \leq \cos x \leq 1$$

$$-e^{-x} \leq e^{-x}\cos x \leq e^{-x}$$

$(e^{-x} > 0)$

$$\lim_{x\to+\infty} -e^{-x} \leq \lim_{x\to+\infty} e^{-x}\cos x \leq \lim_{x\to+\infty} e^{-x}$$

$$0 \leq \lim_{x\to+\infty} \cos x \cdot e^{-x} \leq 0$$

donc $\lim_{x\to+\infty} \cos x\, e^{-x} = 0$

Th. gendarme

⑦ $\lim\limits_{x\to 1} \frac{1}{\ln x} - \frac{1}{x-1} = \lim\limits_{x\to 1} \frac{(x-1)-\ln(x)}{(x-1)\ln x}$

$= \left(\frac{0}{0}\right)$. F.I

$= \lim\limits_{x\to 1} \frac{1-1/x}{\ln x + \frac{x-1}{x}} \left(\frac{0}{0}\right)$ (Hopital)

$\swarrow 0^+ \quad \swarrow 0^+$

R.Hop $\lim\limits_{x\to 1} \frac{-\frac{1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}} = \frac{1}{2}$

⑧ $\lim\limits_{x\to 0} \frac{\sin ax}{\sin bx} = \frac{0}{0}$ F.I l'Hôpital.

$\lim\limits_{x\to 0} \frac{a\cos ax}{b\cos bx} = \frac{a}{b}$.

⑨ $\lim\limits_{x\to +\infty} \frac{\ln \ln x}{\sqrt{x}} = \frac{+\infty}{+\infty}$ F.I

Regle d'Hopital.

$\lim\limits_{x\to +\infty} \frac{\frac{1}{x}\cdot\frac{1}{\ln x}}{\frac{1}{2\sqrt{x}}} = \lim\limits_{x\to +\infty} \frac{\frac{1}{x\ln x}}{1/2\sqrt{x}}$

$= \lim\limits_{x\to +\infty} \frac{2\sqrt{x}}{x\ln x}$

$= \lim\limits_{x\to +\infty} \frac{2}{\sqrt{x}\ln x} = 0$

⑩ On sait que

$\lim\limits_{x\to 0} (1+x)^{1/x} = e$

$\lim\limits_{x\to +\infty} \left(1+\frac{1}{x}\right)^x = e$.

$\left(\frac{x}{x+1}\right) = 1 - \frac{1}{x+1}$

$\left(\frac{x}{x+1}\right)^{2x-1} = \left(1-\frac{1}{x+1}\right)^{2x+1}$

on pose: $-\frac{1}{x+1} = y$.

donc $x = -\frac{1}{y} - 1$

qd $x \to +\infty$ alors $y \to 0$

$\left(1-\frac{1}{x+1}\right)^{2x+1} = (1+y)^{2(-\frac{1}{y}-1)+1}$

$= (1+y)^{-\frac{2}{y}-3}$

$= (1+y)^{-3}(1+y)^{-\frac{2}{y}}$

$= (1+y)^{-3}\left((1+y)^{1/y}\right)^{-2}$

$\lim\limits_{y\to 0} (1+y)^{-3}\left((1+y)^{1/y}\right)^{-2} = \boxed{e^{-2}}$

⑪ $\lim\limits_{x\to 0} (1-2x)^{1/x}$

On pose: $-2x = y \Rightarrow x = -\frac{y}{2}$

et $\frac{1}{x} = -\frac{2}{y}$.

qd $x \to 0$ alors $y \to 0$

$\lim\limits_{x\to 0} (1-2x)^{1/x} = \lim\limits_{y\to 0} (1+y)^{-2/y}$

$= \lim\limits_{y\to 0} \left((1+y)^{1/y}\right)^{-2}$

$= e^{-2} = \frac{1}{e^2}$

(12) $\lim\limits_{x\to 0^+} x^{\sin x}$

$y = x^{\sin x}$

$\ln y = \sin x . \ln x = 0.\infty$

$\ln y = \dfrac{\ln x}{\frac{1}{\sin x}} = \dfrac{-\infty}{\infty}$

on applique la regle de l'Hopital

$\lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{\sin x}} = \lim\limits_{x\to 0^+} \dfrac{1/x}{-\frac{\cos x}{\sin^2 x}}$

$= \lim\limits_{x\to 0^+} \dfrac{1}{x} \times \dfrac{-\sin^2 x}{\cos x}$

$= \lim\limits_{x\to 0} - \dfrac{\sin x}{\cos x} \cdot \dfrac{\sin x}{x}$

$= 0 . 1$

$\lim \ln y = 0$ car $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

donc $\lim y = e^0 = \boxed{1}$

EX04

$$f(x) = \begin{cases} (x-1)^2 & x<3 \\ a & x=3 \\ 4x+b & x>3 \end{cases}$$

$f(3) = a$

$\lim\limits_{\substack{x\to 3 \\ <}} (x-1)^2 = 4$

$\lim\limits_{\substack{x\to 3 \\ >}} 4x+b = 12+b$

f est continue au pt 3

ssi

$\begin{cases} \boxed{a=4} \\ 12+b=4 \Rightarrow \boxed{b=-8} \end{cases}$

$$f(x) = \begin{cases} -2\sin x & x<-\frac{\pi}{2} \\ a\sin x+b\,, & -\frac{\pi}{2}\leq x\leq \frac{\pi}{2} \\ \cos x & x>\frac{\pi}{2} \end{cases}$$

$f(-\frac{\pi}{2}) = a\sin(-\frac{\pi}{2})+b$

$\boxed{f(-\frac{\pi}{2}) = -a+b}$

$\lim\limits_{\substack{x\to -\frac{\pi}{2} \\ <}} f(x) = \lim\limits_{\substack{x\to -\frac{\pi}{2} \\ <}} -2\sin x = 2$

$\lim\limits_{\substack{x\to \frac{\pi}{2} \\ >}} f(x) = \lim\limits_{\substack{x\to \frac{\pi}{2} \\ >}} \cos x = 0$

$f(\frac{\pi}{2}) = a\sin\frac{\pi}{2}+b$

$= a+b.$

f est continue au pts $x=\frac{\pi}{2}$ et $x=-\frac{\pi}{2}$ ssi

$\begin{cases} -a+b=2 \\ a+b=0 \end{cases}$

$2b=2 \Rightarrow \boxed{b=1}$

$a=-b=-1.$

$\boxed{a=-1}$

Exor

Exos!

$$f(x) = \arctan(\sin x^2)$$

$$f'(x) = (\sin(x^2))' \cdot \frac{1}{\sin(x^2)+1}$$

$$= \frac{2x \cdot \cos(x^2)}{1+\sin(x^2)}$$

car:

$$(\arctan(x))' = \frac{1}{1+x^2}.$$

$$f(x) = \arcsin x \, e^{5x}$$

$$f'(x) = \frac{1}{\sqrt{1-x^2}} \cdot e^{5x} + 5e^{5x} \cdot \arcsin x$$

$$(\arcsin)'(x) = \frac{1}{\sqrt{1-x^2}}$$

$$f(x) = \ln(\sqrt{4x-\cos x})$$

$$f'(x) = \frac{(\sqrt{4x-\cos x})'}{\sqrt{4x-\cos x}}$$

$$f'(x) = \frac{\frac{4+\sin x}{2\sqrt{4x-\cos x}}}{\sqrt{4x-\cos x}}$$

$$f'(x) = \frac{4+\sin x}{2(4x-\cos x)}$$

$$f(x) = \frac{(5x+7)^6}{3-x} + \sqrt[3]{x^2+1}$$

$$g(x) = \frac{(5x+7)^6}{3-x}$$

$$g'(x) = \frac{6(5)(5x+7)^5(3-x) + (5x+7)^6}{(3-x)^2}$$

$$= \frac{(5x+7)^5(30(3-x)+5x+7)}{(3-x)^2}$$

$$= \frac{(5x+7)^5(90-30x+5x+7)}{(3-x)^2}$$

$$g'(x) = \frac{(5x+7)^5(97-25x)}{(3-x)^2}$$

$$h(x) = \sqrt[3]{x^2+1} = (x^2+1)^{1/3}$$

$$= e^{1/3 \ln(x^2+1)}$$

$$h'(x) = \left(\frac{1}{3}\ln(x^2+1)\right)' \cdot e^{\frac{1}{3}\ln(x^2+1)}$$

$$h'(x) = \frac{1}{3} \times \frac{2x}{x^2+1} \times e^{1/3 \ln(x^2+1)}$$

$$h'(x) = \frac{2x}{3(x^2+1)} \sqrt[3]{x^2+1}.$$

ou: $h(x) = (x^2+1)^{\frac{1}{3}}$

$$h'(x) = \frac{1}{3}(x^2+1)'(x^2+1)^{\frac{1}{3}-1}$$

$$= \frac{1}{3} \cdot 2x \cdot (x^2+1)^{-\frac{2}{3}}$$

$$h'(x) = \frac{2x}{3} \frac{1}{\sqrt[3]{(x^2+1)^2}}$$