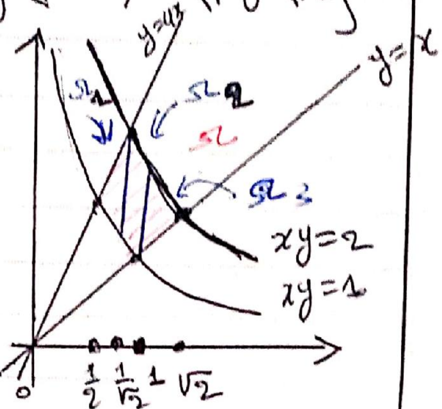
Série d'exercices N° 03
Les intégrales multiples.

Exo 01 : $\iint_{\Omega} f(x,y)\,dx\,dy$ où

- $f(x,y) = x^2y^2$
- $\Omega = \{(x,y) \in \mathbb{R}^2,\ 0 \leq x \leq y \leq 4x,\ 1 \leq xy \leq 2\}$



Alors $\iint_{\Omega} f(x,y)\,dx\,dy = \iint_{\Omega_1} f(x,y)\,dx\,dy + \iint_{\Omega_2} f(x,y)\,dx\,dy + \iint_{\Omega_3} f(x,y)\,dx\,dy$

$$= \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \left(\int_{\frac{1}{x}}^{4x} x^2y^2\,dy \right) dx + \int_{\frac{1}{\sqrt{2}}}^{1} \left(\int_{\frac{1}{x}}^{\frac{2}{x}} x^2y^2\,dy \right) dx$$

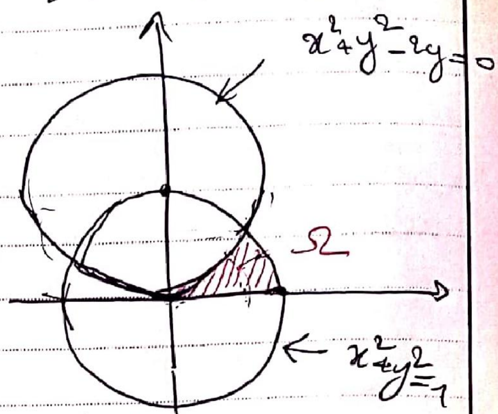
$$+ \int_{1}^{\sqrt{2}} \left(\int_{x}^{\frac{2}{x}} x^2y^2\,dy \right) dx$$

$$= \int_{\frac{1}{2}}^{\frac{1}{\sqrt{2}}} \frac{1}{3}x^2 \left(64x^3 - \frac{1}{x^3} \right) dx + \int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{3}x^2 \left(\frac{8}{x^3} - \frac{1}{x^3} \right) dx$$

$$+ \int_{1}^{\sqrt{2}} \frac{1}{3}x^2 \cdot \left(\frac{8}{x^3} - x^3 \right) dx = \frac{7}{3}\cdot \log 2$$

$\iint_\Omega \sqrt{x^2+y^2}\, dx\,dy$ où

$\Omega = \{(x,y) \in \mathbb{R}^2,\ x \geq 0,\ y \geq 0\ \ 2y \leq x^2+y^2 \leq 1\}$



On pose $x = r\cos\theta$, $y = r\sin\theta$

$\theta \in [0, \frac{\pi}{6}]$, $r \in [2\sin\theta, 1]$

$$\iint_\Omega \sqrt{x^2+y^2}\, dx\,dy = \int_0^{\frac{\pi}{6}} \int_{2\sin\theta}^{1} r.r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{6}} \frac{1}{3} r^3 \Big|_{2\sin\theta}^{1} = \int_0^{\frac{\pi}{6}} \left(\frac{1}{3} - \frac{8}{3}\sin^3\theta\right) d\theta$$

$$\frac{\pi}{18} - \frac{8}{3}\int_0^{\frac{\pi}{6}} \sin\theta(1-\cos^2\theta)\, d\theta$$

$$= \frac{\pi}{18} - \frac{8}{3}\int_0^{\frac{\pi}{6}} \sin\theta + \frac{8}{3}\int_0^{\frac{\pi}{6}} \cos^2\theta . \sin\theta\, d\theta$$

$$= \frac{\pi}{18} + \frac{8}{3}(\cos\theta)\Big|_0^{\frac{\pi}{6}} - \frac{8}{3} . \frac{1}{3}\cos^3\theta \Big|_0^{\frac{\pi}{6}}$$

$$\frac{\pi}{18} + \frac{8}{3}.\left(\frac{\sqrt{3}}{2} - 1\right) - \frac{8}{3} . \frac{1}{3}\left(\left(\frac{\sqrt{3}}{2}\right)^3 - 1\right)$$

$$\frac{\pi}{18} - \frac{16}{9} + \sqrt{3}$$

•• $\iint_{\Omega} f(x,y)\,dx\,dy$ où

$\Omega = \{(x,y) \in \mathbb{R}^2,\ x \geq 0,\ y \geq 0,\ 9x^2+4y^2 \leq 36\}$

$f(x,y) = x^2 y^4\, dx\, dy$;

En utilisant les coordonnés polaires, le domaine Ω peut être décrit par

$$\begin{cases} x = 2r\cos\theta \\ y = 3r\sin\theta \end{cases} \quad \text{avec } 0 \leq r \leq 1,\ 0 \leq \theta \leq \frac{\pi}{2} ;$$

alors $\iint_{\Omega} f(x,y)\,dx\,dy = \int_0^{\frac{\pi}{2}} \left(\int_0^1 (4r^2\cos^2\theta)(81r^4\sin^4\theta)\, 6r\,dr \right) d\theta$

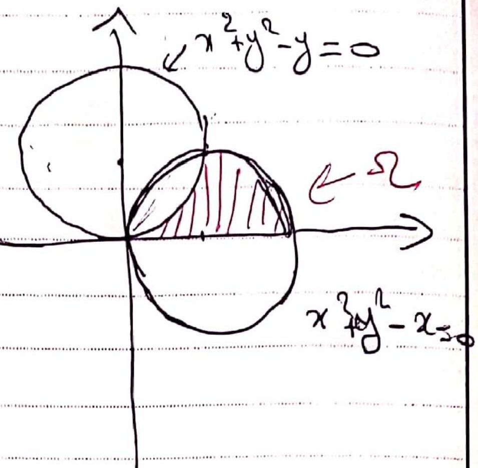
$$= 243 \int_0^{\frac{\pi}{2}} \sin^4\theta \cos^2\theta\, d\theta = 243 \cdot \frac{\pi}{32}.$$

Exo 2 : $\iint_\Omega (x^2+y^2+2xy)\,dx\,dy$

$\Omega = \{(x,y) \in \mathbb{R}^2,\ y \geq 0,\ x^2+y^2-y \geq 0,\ x^2+y^2-x \leq 0\}$

On pose $x = r\cos\theta$, $y = r\sin\theta$

$\theta \in [0, \frac{\pi}{4}]$, $r \in [\sin\theta, \cos\theta]$



$\iint_\Omega f(x,y)\,dx\,dy$

$$= \int_0^{\frac{\pi}{4}} \int_{\sin\theta}^{\cos\theta} (r^2 + 2r^2\sin\theta\cos\theta)\, r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\int_{\sin\theta}^{\cos\theta} r^3 (1 + 2\sin\theta\cos\theta)\, dr \right) d\theta$$

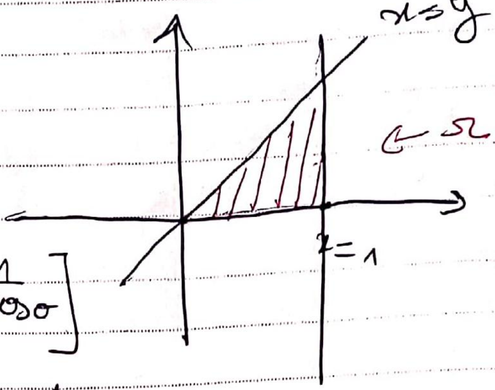
$$= \int_0^{\frac{\pi}{4}} \frac{1}{4}(\cos^4\theta - \sin^4\theta)(1+2\sin\theta\cos\theta)\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\cos 2\theta}{4}(1+\sin 2\theta)\, d\theta$$

$$= \frac{1}{8}\int_0^{\frac{\pi}{4}} 2\cos 2\theta \cdot (1+\sin 2\theta)\, d\theta = \int f' f$$

$$= \frac{1}{8} \cdot \frac{1}{2}(1+\sin 2\theta)^2 \Big|_0^{\frac{\pi}{4}} = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$

Exo 03 : $\iint_{\Omega} \sqrt{x^2+y^2}$, $\Omega = \{(x,y) \in \mathbb{R}^2 \; 0<y<x<1\}$



on pose $x = r\cos\theta$
$y = r\sin\theta$

$\theta \in [0, \frac{\pi}{4}]$ $r \in [0, \frac{1}{\cos\theta}]$

$$\iint_{\Omega} f(x,y)\,dx\,dy = \int_0^{\frac{\pi}{4}} \int_0^{\frac{1}{\cos\theta}} r^2\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{3} r^3 \Big|_0^{\frac{1}{\cos\theta}} \right) d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{3} \frac{1}{\cos^3\theta}\,d\theta$$

$$= \frac{1}{3} \int_0^{\frac{\pi}{4}} \frac{\cos\theta}{\cos^4\theta}\,d\theta = \frac{1}{3} \int_0^{\frac{\pi}{4}} \frac{\cos\theta}{(1-\sin^2\theta)^2}\,d\theta \quad (*)$$

$u = \sin\theta \Rightarrow du = \cos\theta\,d\theta \Rightarrow \int \frac{\cos\theta}{1-\sin^2\theta}\,d\theta$

$$= \cancel{\frac{1}{3}} \int \frac{du}{(1-u^2)^2} = \int \frac{1}{4} \left[\frac{1}{(1-u)^2} + \frac{1}{(1+u)^2} \right] + \frac{1}{2} \left[\frac{1}{1-u^2} \right] du$$

$$= \frac{1}{4} \left[\frac{1}{1-u} - \frac{1}{1+u} \right] + \frac{1}{4} \ln \frac{1+u}{1-u}$$

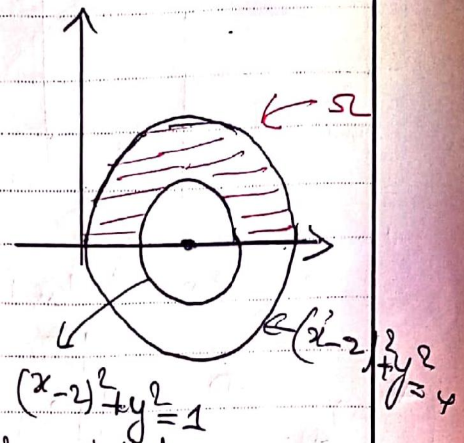
$$\Rightarrow (*) = \frac{1}{12} \left[\frac{1}{1-\sin\theta} - \frac{1}{1+\sin\theta} + \ln \frac{1+\sin\theta}{1-\sin\theta} \right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{12} \left[\frac{1}{1-\frac{\sqrt{2}}{2}} + \frac{1}{1+\frac{\sqrt{2}}{2}} + \ln \frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}} \right]$$

Exo : 04 $\iint_\Omega f(x,y)\,dx\,dy$ où

- $f(x,y) = \cos(x^2+y^2-4x+4)\,dx\,dy$
- $\Omega = \{(x,y) \in \mathbb{R}^2,\ y \geq 0,\ 1 \leq (x-2)^2+y^2 \leq 4\}$

On pose $\begin{cases} x = 2 + r\cos\theta \\ y = r\sin\theta \end{cases}$



$1 \leq r \leq 2, \quad 0 \leq \theta \leq \pi$

alors $\iint_\Omega f(x,y)\,dx\,dy = \iint_\Omega \cos(x^2+y^2-4x+4)\,dx\,dy$

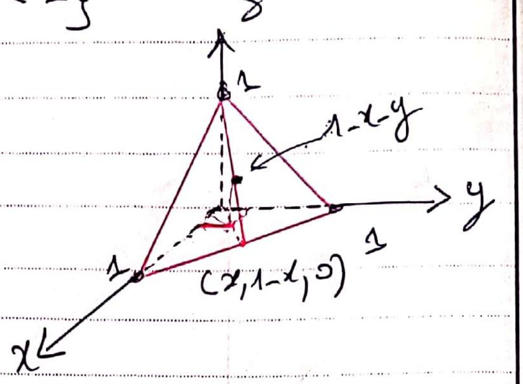
$= \iint_\Omega \cos\left((x-2)^2+y^2\right)\,dx\,dy$

$= \int_0^\pi \left(\int_1^2 r\cos r^2\,dr \right) d\theta = \frac{\pi}{2}(\sin 4 - \sin 1)$

Car $\int f' \cos f = \sin f$

Exo 08 : • $\Omega = \{(x,y,z) \in \mathbb{R}^3,\ x \geq 0,\ y \geq 0,\ z \geq 0,\ x+y+z \leq 1\}$

1) $\displaystyle\iiint_\Omega \frac{dx\,dy\,dz}{(x+y+z+1)^2}$



$$\iiint_\Omega \frac{dx\,dy\,dz}{(x+y+z+1)^2} = \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} \frac{dz}{(x+y+z+1)^2} \right) dy \right) dx$$

$$= \int_0^1 \left(\int_0^{1-x} \left(-\frac{1}{2} + \frac{1}{x+y+1} \right) dy \right) dx$$

$$= \int_0^1 \left(-\frac{1}{2} + \frac{x}{2} + \log 2 - \log(1+x) \right) dx$$

$$= \frac{3}{4} - \log 2 .$$

•• $\Omega = \{(x,y,z) \in \mathbb{R}^3,\ 1 \leq x^2+y^2+z^2 \leq 4\}$

$\displaystyle\iiint_\Omega \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$, on utilise les coordonnées sphériques on a : $\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$

$1 \leq r \leq 2,\ 0 \leq \theta \leq \pi,\ 0 \leq \varphi \leq 2\pi$

alors $\iiint_{\Omega} \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}} = \int_0^{2\pi}\left(\int_0^{\pi}\left(\int_1^{2} r \sin\theta\, dr\right)d\theta\right)d\varphi$

$= 6\pi$.

$\Omega = \{(x,y,z) \in \mathbb{R}^3,\ x^2+y^2+z^2 \leq 1\}$

$$\iiint_{\Omega} \frac{dx\,dy\,dz}{\sqrt{(x-2)^2+y^2+z^2}} = \int_{-1}^{1}\left(\iint_{B(0,\sqrt{1-x^2})} \frac{dy\,dz}{\sqrt{(x-2)^2+y^2+z^2}}\right)dx$$

$$= \int_{-1}^{1}\left(\int_0^{2\pi}\left(\int_0^{\sqrt{1-x^2}} \frac{1}{\sqrt{(x-2)^2+r^2}}\, r\,dr\right)d\theta\right)dx$$

$$= 2\pi \int_{-1}^{1} \left(\sqrt{5-4x} + x - 2\right)dx = \frac{2\pi}{3}$$

$\Omega = \{(x,y,z) \in \mathbb{R}^3,\ z \geq 0,\ x^2+y^2 \leq 1,\ x^2+y^2+z^2 \leq 4\}$

$$\iiint_{\Omega} z\,dx\,dy\,dz = \iint_{B(0,1)}\left(\int_0^{\sqrt{4-x^2-y^2}} z\,dz\right)dx\,dy$$

$$= \frac{1}{2}\iint_{B(0,1)} (4-x^2-y^2)\,dx\,dy$$

$$= \frac{1}{2}\int_0^1\left(\int_0^{2\pi}(4-r^2)\,r\,d\theta\right)dr = \pi\int_0^1 (4-r^2)\,r\,dr$$

$$= \frac{7\pi}{4}$$